Overview of IUT2

Wojciech Porowski

University of Nottingham

September 2021

Introduction

This is a continuation of the previous talk, titled "Overview of Étale Theta Function". We use the same notation and references.
In this talk we will discuss the notion of mono-theta environment, cyclotomic rigidity isomorphism, multiradiality of theta monoids, and graph-theoretic evaluation of theta function.

Mono-theta environment, ([EtTh],§2, 2.13), ([IUT2], §1)

For a group $\Pi \twoheadrightarrow G_K$ we write $\Pi[\mu_n] = \mu_n \rtimes \Pi$.
Model (mod n) mono-theta environment = a triple consisting of:
(a) the topological group $\Pi^{tp}_{\underline{\underline{Y}}}[\mu_n]$,
(b) a subgroup $\mathcal{D}_{\underline{\underline{Y}}} \subset \mathrm{Out}(\Pi^{tp}_{\underline{\underline{Y}}}[\mu_n])$,
(c) μ_n-conjugacy class of a subgroup $Im(s^{\Theta}_{\underline{\underline{\ddot{Y}}}}) \subset \Pi^{tp}_{\underline{\underline{Y}}}[\mu_n]$.
We need to define subgroups appearing in (b) and (c). For (b), consider two maps:

$$K^* \to H^1(\Pi^{tp}_{\underline{\underline{Y}}}, \mu_n) \to \mathrm{Out}(\Pi^{tp}_{\underline{\underline{Y}}}[\mu_n])$$

and

$$\mathrm{Gal}(\underline{\underline{Y}}/\underline{\underline{X}}) \to \mathrm{Out}(\Pi^{tp}_{\underline{\underline{Y}}}[\mu_n]).$$

We define $\mathcal{D}_{\underline{\underline{Y}}}$ to be a subgroup of $\mathrm{Out}(\Pi^{tp}_{\underline{\underline{Y}}}[\mu_n])$ generated by K^* and $\mathrm{Gal}(\underline{\underline{Y}}/\underline{\underline{X}})$ using the above maps.

Mono-theta environment (2)

To define the subgroup in (c), start with the s.e.s:

$$1 \to \mu_n \to \mu_n \rtimes \Pi \to \Pi \to 1,$$

thus we have tautological section $s^{alg} : \Pi \ni \pi \mapsto (1, \pi) \in \mu_n \rtimes \Pi$.
Recall that, (relative to s^{alg})

$$\{\text{sections up to } \mu_n - \text{conj.}\} \equiv H^1(\Pi, \mu_n).$$

We apply this to $1 \to \mu_n \to \mu_n \rtimes \Pi^{tp}_{\underline{\underline{\ddot{Y}}}} \to \Pi^{tp}_{\underline{\underline{\ddot{Y}}}} \to 1$ and to the cohomology class

$$\underline{\underline{\ddot{\eta}}} \in H^1(\Pi^{tp}_{\underline{\underline{\ddot{Y}}}}, \mu_n)$$

corresponding to theta function.
$\Rightarrow$ we obtain a conjugacy class of sections $s^{\Theta} : \Pi^{tp}_{\underline{\underline{\ddot{Y}}}} \to \Pi^{tp}_{\underline{\underline{\ddot{Y}}}}[\mu_n]$.

Mono-theta environment (3)

Composing with the inclusion $\Pi^{tp}_{\underline{\underline{\ddot{Y}}}}[\mu_n] \hookrightarrow \Pi^{tp}_{\underline{\underline{Y}}}[\mu_n]$ we obtain

$$s^{\Theta}_{\underline{\underline{\ddot{Y}}}} \colon \Pi^{tp}_{\underline{\underline{\ddot{Y}}}} \to \Pi^{tp}_{\underline{\underline{Y}}}[\mu_n],$$

which we call a theta section. The image of theta section defines the subgroup in (c).
Summarizing, a model (mod N) mono-theta environment is a triple

$$(\Pi^{tp}_{\underline{\underline{Y}}}[\mu_n],\ \mathcal{D}_{\underline{\underline{Y}}},\ \mu_n - \text{conjugacy class of } Im(s^{\Theta}_{\underline{\underline{\ddot{Y}}}})).$$

Mono-theta environment (4)

At this point, we need the following lemma.

Lemma (EtTh, §2, 2.11)

The kernel of the surjection $\Pi^{tp}_{\underline{\underline{Y}}}[\mu_n] \twoheadrightarrow \Pi^{tp}_{\underline{\underline{Y}}}$ (i.e. the group μ_n) is equal to the union of the centralizers of all open subgroups of $\Pi^{tp}_{\underline{\underline{Y}}}[\mu_n]$.

In particular, the quotient $\Pi^{tp}_{\underline{\underline{Y}}}[\mu_n] \twoheadrightarrow \Pi^{tp}_{\underline{\underline{Y}}}$ can be constructed group theoretically.

Then a mono-theta environment $\mathbb{M}^{\Theta}$ is any triple

$$(\Pi, \mathcal{D}, \mu_n - \text{conjugacy class of } s^{\Theta}_{\Pi}),$$

isomorphic to the model mono-theta environment.

Properties of mono-theta environment

Fix a mono-theta environment $\mathbb{M}$, so $\Pi \cong \Pi^{tp}_{\underline{\underline{Y}}}[\mu_n]$.
From $\mathbb{M}$ we will construct various subquotients of Π - they will be denoted with brackets "$(\mathbb{M})$".

- From the previous lemma we construct the quotient $\Pi \twoheadrightarrow \Pi^{tp}_{\underline{\underline{Y}}}(\mathbb{M})$ (thus $\Pi^{tp}_{\underline{\underline{Y}}}(\mathbb{M}) \cong \Pi^{tp}_{\underline{\underline{Y}}}$).
- By using the $\mathcal{D}$ portion of $\mathbb{M}$ we construct the group $\Pi^{tp}_{\underline{\underline{X}}}(\mathbb{M})$ (isomorphic to $\Pi^{tp}_{\underline{\underline{X}}}$) such that we have the inclusion $\Pi \twoheadrightarrow \Pi^{tp}_{\underline{\underline{Y}}}(\mathbb{M}) \subset \Pi^{tp}_{\underline{\underline{X}}}(\mathbb{M})$.
- Therefore, using anabelian results concerning the group $\Pi^{tp}_{\underline{\underline{X}}}$ we may further construct fundamental groups off all covers of X from our main geometric diagram.

Properties of mono-theta environment (2)

Next, we will use the s^{Θ}_{Π}-portion of $\mathbb{M}$.
Recall that the cohomology class of theta function comes from the quotient $(\Pi^{tp}_{\underline{\underline{\ddot{Y}}}})^{\Theta}$. Thus, the restriction of the section s^{Θ}_{Π} to the kernel of the surjection

$$\Pi^{tp}_{\underline{\underline{Y}}}(\mathbb{M}) \twoheadrightarrow (\Pi^{tp}_{\underline{\underline{Y}}})^{\Theta}(\mathbb{M})$$

coincides with the tautological section s^{alg}.
In particular, by taking quotient of Π by the image of this restriction we construct the group

$$\Pi|_{(\Pi^{tp}_{\underline{\underline{Y}}})^{\Theta}(\mathbb{M})}$$

(isomorphic to $(\Pi^{tp}_{\underline{\underline{Y}}})^{\Theta}[\mu_n]$).

Properties of mono-theta environment (3)

Then, by considering preimages of the following subgroups

$$\ell\Delta_{\Theta}(\mathbb{M}) \subset (\Delta^{tp}_{\underline{\underline{Y}}})^{\Theta}(\mathbb{M}) \subset (\Pi^{tp}_{\underline{\underline{Y}}})^{\Theta}(\mathbb{M})$$

along the surjection

$$\Pi|_{(\Pi^{tp}_{\underline{\underline{Y}}})^{\Theta}(\mathbb{M})} \twoheadrightarrow (\Pi^{tp}_{\underline{\underline{Y}}})^{\Theta}(\mathbb{M})$$

we obtain two subquotients of Π

$$\Pi|_{\ell\Delta_{\Theta}(\mathbb{M})} \subset \Pi|_{(\Delta^{tp}_{\underline{\underline{Y}}})^{\Theta}(\mathbb{M})}$$

(isomorphic to $\ell\Delta_{\Theta}[\mu_n] \subset (\Delta^{tp}_{\underline{\underline{Y}}})^{\Theta}[\mu_n]$).

Properties of mono-theta environment (4)

Write $\Pi_\mu(\mathbb{M})$ for the subgroup of Π corresponding to μ_n.
Thus the subquotient $\Pi|_{\ell\Delta_\Theta(\mathbb{M})}$ fits naturally in the short exact sequence

$$1 \to \Pi_\mu(\mathbb{M}) \to \Pi|_{\ell\Delta_\Theta(\mathbb{M})} \to \ell\Delta_\Theta(\mathbb{M}) \to 1$$

Observe that the "theta section portion" s_Π^Θ of the mono-theta environment $\mathbb{M}$ determines a splitting of this exact sequence.
We are going to construct another splitting of this sequence, corresponding to the tautological section s^{alg}.

Cyclotomic rigidity ([EtTh], §2, 2.19), ([IUT2], §1)

Recall that the "$\mathcal{D}$-portion" of a (model) mono-theta environment is a subgroup of $\mathrm{Out}(\Pi^{tp}_{\underline{\underline{Y}}}[\mu_n])$ "generated" by $\Pi^{tp}_{\underline{\underline{X}}}$ and $K^{\times}$.
Observe that by restricting this outer action to the group $\Pi|_{(\Delta^{tp}_{\underline{\underline{Y}}})^{\Theta}(\mathbb{M})}$ it factors through an action of

$$\Delta^{tp}_{\underline{\underline{X}}}/\Delta^{tp}_{\underline{\underline{Y}}} \cong (\Delta^{tp}_{\underline{\underline{X}}})^{\Theta}/(\Delta^{tp}_{\underline{\underline{Y}}})^{\Theta} \cong \ell\mathbb{Z}$$

(because $\Pi|_{(\Delta^{tp}_{\underline{\underline{Y}}})^{\Theta}(\mathbb{M})}$ is abelian).
Consider a subset S of $\Pi|_{\ell\Delta_{\Theta}(\mathbb{M})}$ consisting of elements of the form $\lambda(a)a^{-1}$, where $\lambda \in (\Delta^{tp}_{\underline{\underline{X}}})^{\Theta}/(\Delta^{tp}_{\underline{\underline{Y}}})^{\Theta}$ and $a \in \Pi|_{(\Delta^{tp}_{\underline{\underline{Y}}})^{\Theta}(\mathbb{M})}$.

Cyclotomic rigidity (2)

Lemma

The subset S is equal to the image of a tautological section s^{alg}

Proof.

Recall that $\ell\Delta_\Theta \subset (\Delta^{tp}_{\underline{\underline{Y}}})^\Theta \subset (\Delta^{tp}_{\underline{\underline{X}}})^\Theta$ is isomorphic to:

$$\begin{bmatrix} 1 & 0 & \ell\widehat{\mathbb{Z}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \hookrightarrow \begin{bmatrix} 1 & \widehat{\mathbb{Z}} & \ell\widehat{\mathbb{Z}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \hookrightarrow \begin{bmatrix} 1 & \widehat{\mathbb{Z}} & \ell\widehat{\mathbb{Z}} \\ 0 & 1 & \ell\widehat{\mathbb{Z}} \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the lemma follows from $[(\Delta^{tp}_{\underline{\underline{X}}})^\Theta, (\Delta^{tp}_{\underline{\underline{X}}})^\Theta] = \ell\Delta_\Theta$ (commutator property of theta group). $\square$

Cyclotomic rigidity (3)

In other words, we obtain a section s^{alg}:

$$\begin{tikzcd}
 & & S \arrow[d, hook] \arrow[dr, "\cong"] & & \\
1 \arrow[r] & \Pi_\mu(\mathbb{M}) \arrow[r] & \Pi|_{\ell\Delta_\Theta(\mathbb{M})} \arrow[r] & \ell\Delta_\Theta(\mathbb{M}) \arrow[r] & 1.
\end{tikzcd}$$

By taking the difference between s^{Θ}_{Π} and s^{alg} we obtain a cohomology class in

$$H^1(\ell\Delta_\Theta(\mathbb{M}), \Pi_\mu(\mathbb{M})) = \mathrm{Hom}(\ell\Delta_\Theta(\mathbb{M}), \Pi_\mu(\mathbb{M})).$$

This homomorphism induces an isomorphism,

$$\ell\Delta_\Theta(\mathbb{M}) \otimes \mathbb{Z}/n\mathbb{Z} \cong \Pi_\mu(\mathbb{M})$$

called the cyclotomic rigidity isomorphism.

Cyclotomic rigidity (4)

When $\mathbb{M}$ is a model mono-theta environment, then cyclotomic rigidity coincides with an isomorphism $\ell\Delta_\Theta \otimes \mathbb{Z}/n\mathbb{Z} \cong \mu_n$ induced from the natural isomorphism $\ell\Delta_\Theta \cong \widehat{\mathbb{Z}}(1)$.
Observe that it is also possible to construct a mono-theta environment from the topological group $\Pi^{tp}_{\underline{\underline{X}}}$, by taking Π_μ to be $\ell\Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z}$ and following the construction from the first talk.
Then, the cyclotomic rigidity will be simply the identity.

Summary

To summarise:

- A (mod n) mono-theta environment $\mathbb{M}$ is an "amalgamation" of certain fundamental group with (an isomorph of) the group of roots of unity μ_n.
- From $\mathbb{M}$, we may construct two subquotients $\ell\Delta_\Theta(\mathbb{M}) \otimes \mathbb{Z}/n\mathbb{Z}$ and $\Pi_\mu(\mathbb{M})$ (both isomorphic to $\mathbb{Z}/n\mathbb{Z}(1)$) together with a canonical isomorphism between them (cyc. rig.).
- In particular, cyclotomic rigidity is compatible with every isomorphism $\mathbb{M} \cong \mathbb{M}'$ of mono-theta environments.

$\mathbb{F}_\ell^{\rtimes\pm}$ symmetry

We remark that mono-theta environment $\mathbb{M} = (\Pi, \mathcal{D}, s_\Pi^\Theta)$ has another type of symmetry. Recall that we have the group

$$\Pi_C(\mathbb{M})/\Pi_{\underline{X}}(\mathbb{M}) \cong \Delta_C(\mathbb{M})/\Delta_{\underline{X}}(\mathbb{M}) \cong \mathbb{F}_\ell^{\rtimes\pm}$$

Moreover, there is a natural action of $\Pi_C(\mathbb{M})$ on the group $\Pi|_{\ell\Delta_\Theta(\mathbb{M})}$. Then, this action preserves the image of the tautological section s^{alg}, in particular the cyclotomic rigidity isomorphism is compatible with the $\mathbb{F}_\ell^{\rtimes\pm}$-symmetry.

Further remarks

- The construction of a mono-theta environment does not depend (up to an isomorphism) on the choice of of a theta function inside its $\ell\mathbb{Z} \times \mu_2$-orbit (shifting automorphisms).
- Write $\mathbb{M}_*$ for the inverse limit of (mod N) mono-theta environments (over $N \in \mathbb{N}$)

  $$\{\ldots \to \mathbb{M}_{N'} \to \mathbb{M}_N \to \ldots\}$$

  Also, write $\Pi_\mu(\mathbb{M}_*) = \lim_N \Pi_\mu(\mathbb{M}_N)$ $(\cong \widehat{\mathbb{Z}}(1))$.
  Then, from $\mathbb{M}_*$ we may construct an isomorphism

  $$\ell\Delta_\Theta(\mathbb{M}_*) \cong \Pi_\mu(\mathbb{M}_*),$$

  also called the cyclotomic rigidity isomorphism.
- There is another construction of a mono-theta environment, from a certain tempered Frobenioid, which we now briefly discuss.

Frobenioids

We recall a working definition of a Frobenioid.
Let $\mathcal{D}$ be a base category (usually a category of étale or tempered covers of some base scheme).
$\Phi =$ a presheaf of monoids on $\mathcal{D}$ ("divisor monoid").
$B =$ be a presheaf of abelian groups on $\mathcal{D}$ ("rational functions").
$div\colon B \to \Phi^{gp} =$ morphism of functors ("divisor map").
Define a fibered category $\mathcal{D}' \to \mathcal{D}$:
For $A \in Obj(\mathcal{D})$, we have $Obj(\mathcal{D}'_A) = \Phi^{gp}(A)$ ("line bundle with a section")
For $a, a' \in \mathcal{D}'$, morphisms $Mor(a, a') =$ set of pairs $(d, f) \in \Phi(A) \times B(A)$ satisfying $a + d = a' + div(f)$.

Frobeniods (2)

Finally, we define a fibered category $\mathcal{F} \to \mathcal{D}$:
On objects, $Obj(\mathcal{F}) = Obj(\mathcal{D}')$.
For A in D and $a, b \in Obj(\mathcal{F}_A)$ we have

$$Mor_{\mathcal{F}}(a, b) = \{\varphi \in Mor_{D'_A}(a^{\otimes n}, b),\ n \in \mathbb{N}\}$$

The category $\mathcal{F}$ is a Frobenioid associated to the data Φ, B, div.
("category of line bundles on $\mathcal{D}$")

Examples of Frobenioids

A simple example:
K = p-adic local field, G_K = absolute Galois group, $\mathcal{D} = \mathcal{B}(G_K)$;
For L/K define $\Phi(L) = \mathcal{O}_L^{\triangleright}/\mathcal{O}_L^{\times} \cong \mathbb{N}$, $B(L) = L^{\times}$, $div(L) = v_L$.
We obtain a p-adic Frobenioid, equivalent to the data $G_K \curvearrowright \mathcal{O}^{\triangleright}_{K^{\mathrm{alg}}}$.
Tempered Frobenioid: $\underline{\underline{\mathcal{F}}}$
Here the base category is given by tempered covers of $\underline{\underline{X}}$ $(= \mathcal{B}(\Pi_{\underline{\underline{X}}}))$
We avoid giving a precise definition of the corresponding functors Φ and B.
Instead, we will list some properties of the Frobenioid $\underline{\underline{\mathcal{F}}}$.

Tempered Frobenioid ([EtTh], §3-§5)

Properties of $\underline{\underline{\mathcal{F}}}$:

- Functors Φ and B are rich enough to allow the construction of a theta function, as a cohomology class in $H^1(\Pi_{\ddot{Y}}, \mu_n)$, where μ_n is a subgroup of n-torsion elements of $B(\ddot{Y})$.
- If fact, theta function is constructed as a quotient of two sections of a certain line bundle.
- More generally we may construct (split) theta monoids $\mathcal{O}^\times \times \Theta^{\mathbb{N}}$, note that n-torsion of $\mathcal{O}^\times$ is identified with μ_n above.
- Furthermore, we may also construct various mono-theta environments $\mathbb{M}_N^\Theta = \mathbb{M}_N^\Theta(\underline{\underline{\mathcal{F}}})$ whose exterior cyclotomes $\Pi_\mu(\mathbb{M}_N^\Theta)$ are identified with torsion groups μ_n. Thus, the exterior cyclotome represents a Frobenius-like part of the mono-theta environment $\mathbb{M}_N^\Theta(\underline{\underline{\mathcal{F}}})$.

Theta monoid ([IUT2] §3, 3.1)

Let $\mathbb{M}_*$ be a projective system of mono-theta environments.
We also introduce the notation:

$$_\infty H^1(G, A) = colim_{H \subset G, open+fin.ind.} H^1(H, A).$$

For every inversion automorphism ι of $\underline{\underline{\ddot{Y}}}$ (i.e. irreducible component P_0), we have the following objects, constructed from $\mathbb{M}_*$.
An ℓth root of an inverse of theta function, defined up to $\mu_{2\ell}$

$$\underline{\underline{\theta}}^\iota_{env}(\mathbb{M}_*) \subset H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}_*), \Pi_\mu(\mathbb{M}_*)).$$

All "roots" of $\underline{\underline{\theta}}^\iota_{env}(\mathbb{M}_*)$, defined up to torsion

$$_\infty\underline{\underline{\theta}}^\iota_{env}(\mathbb{M}_*) \subset {}_\infty H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}_*), \Pi_\mu(\mathbb{M}_*)).$$

Theta monoid (2)

Moreover, we have a group isomorphic to $\mathcal{O}^{\times}_{K^{\mathrm{alg}}}$:

$$\mathcal{O}^{\times}(\mathbb{M}_*) \subset {}_{\infty}H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}_*), \Pi_{\mu}(\mathbb{M}_*)).$$

We define the following monoids

$$\Psi^{\iota}_{env}(\mathbb{M}_*) = \mathcal{O}^{\times} \cdot \underline{\underline{\theta}}^{\iota}_{env}(\mathbb{M}_*)^{\mathbb{N}} \subset {}_{\infty}H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}_*), \Pi_{\mu}(\mathbb{M}_*)),$$

as well as the ∞-version

$${}_{\infty}\Psi^{\iota}_{env}(\mathbb{M}_*) = \mathcal{O}^{\times} \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota}_{env}(\mathbb{M}_*)^{\mathbb{N}} \subset {}_{\infty}H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}_*), \Pi_{\mu}(\mathbb{M}_*)).$$

These monoids are called theta monoids. We also take the collection of theta monoids obtained with respect to all inversion automorphisms ι

$$\Psi_{env}(\mathbb{M}_*) = \{\Psi^{\iota}_{env}(\mathbb{M}_*)\}_{\iota}, \quad {}_{\infty}\Psi_{env}(\mathbb{M}_*) = \{{}_{\infty}\Psi^{\iota}_{env}(\mathbb{M}_*)\}_{\iota}.$$

Theta monoid (3)

We now go back briefly to the tempered Frobenioid $\underline{\underline{\mathcal{F}}}$. From the category $\underline{\underline{\mathcal{F}}}$ one can construct (from various functors to groups/monoids in the definition of a Frobenioid) the following monoids

$$\Psi_{\mathcal{F}^{\Theta},\alpha}, \quad {}_{\infty}\Psi_{\mathcal{F}^{\Theta},\alpha}, \quad \Psi_{\mathcal{C}}$$

corresponding respectively to "$\mathcal{O}^{\times} \cdot (\underline{\underline{\theta}}^{\iota})^{\mathbb{N}}$", "$(\mathcal{O}^{\times} \cdot \underline{\underline{\theta}}^{\iota})^{\mathbb{Q}_{\geq 0}}$" (note that "Frobenius-like functions" are obtained as quotients of sections of line bundles), as well as to the "monoid of constant functions" $\mathcal{O}^{\triangleright}_{K^{\mathrm{alg}}}$ (the index α plays the role of an automorphism ι).

Furthermore, all these monoids are equipped with an action of a topological group isomorphic to $\Pi^{tp}_{\underline{\underline{X}}}$.

Multiradiality of theta monoid ([IUT2], §3, 3.4)

Recall that from the tempered Frobenioid we may construct a mono-theta environment $\mathbb{M}_* = \mathbb{M}_*(\underline{\underline{\mathcal{F}}})$. Then, by forming Kummer classes of elements from $\Psi_{\mathcal{F}^{\Theta},\alpha}$ and $_\infty\Psi_{\mathcal{F}^{\Theta},\alpha}$ we obtain Kummer isomorphisms

$$\Psi_{\mathcal{F}^{\Theta},\alpha} \cong \Psi^{\iota}_{env}(\mathbb{M}_*), \quad {}_\infty\Psi_{\mathcal{F}^{\Theta},\alpha} \cong {}_\infty\Psi^{\iota}_{env}(\mathbb{M}_*)$$

with respect to some bijection $\alpha \leftrightarrow \iota$ (Frobenius-like to mono-theta). Moreover, since $\mathbb{M}_* \cong \mathbb{M}_*(\Pi^{tp}_{\underline{\underline{X}}})$, by applying our results about mono-theta environment we obtain isomorphisms

$$\Psi^{\iota}_{env}(\mathbb{M}_*) \cong \Psi^{\iota}_{env}(\mathbb{M}_*(\Pi^{tp}_{\underline{\underline{X}}})), \quad {}_\infty\Psi^{\iota}_{env}(\mathbb{M}_*) \cong {}_\infty\Psi^{\iota}_{env}(\mathbb{M}_*(\Pi^{tp}_{\underline{\underline{X}}}))$$

(mono-theta to étale-like).

Multiradiality of theta monoid (2)

Thus, we obtain an algorithmic construction that links Frobenius-like theta monoid to the étale-like theta monoid through the mono-theta environment. Then, the cyclotomic rigidity of mono-theta environment implies the following property, also called multiradiality

Multiradiality of theta monoid

The algorithm that constructs isomorphisms between Frobenius-like and étale-like theta monoids

$$\Psi_{\mathcal{F}^{\Theta},\alpha} \cong \Psi^{\iota}_{env}(\mathbb{M}_*(\Pi^{tp}_{\underline{\underline{X}}})), \quad {}_{\infty}\Psi_{\mathcal{F}^{\Theta},\alpha} \cong {}_{\infty}\Psi^{\iota}_{env}(\mathbb{M}_*(\Pi^{tp}_{\underline{\underline{X}}}))$$

is compatible with automorphisms of the Galois monoid $G_K \curvearrowright \Psi_{\mathcal{C}}^{\times\mu}$ (i.e., $G_K \curvearrowright \mathcal{O}^{\times\mu}$; recall that $\mathcal{O}^{\times\mu} = \mathcal{O}^{\times}/\mu$).

Multiradiality of theta monoid (3)

Note that the point of the above statement (i.e., multiradiality) is not about the existence of an isomorphism between two monoids (as this is trivial), it is about the existence of an algorithm which agrees with Kummer theory and is compatible with automorphisms of $G_K \curvearrowright \mathcal{O}^{\times\mu}$. Moreover, observe that we also have splittings of theta monoids, i.e.,

$$\mathcal{O}^{\times} \cdot (\underline{\underline{\theta}}^{\iota})^{\mathbb{N}} \cong \mathcal{O}^{\times} \times (\underline{\underline{\theta}}^{\iota})^{\mathbb{N}}$$

(well-defined up to a root of unity) obtained by restricting cohoomology classes to the decomposition group of $\mu_{-}(\underline{\underline{\ddot{Y}}})$ (similarly for the ∞-version). Therefore, these splittings (up to roots of unity) are also multiradial.

Notation

From now on we will switch from considering purely local situation to a local situation at a place $\underline{v} \in \underline{\mathbb{V}}$. Therefore our notation changes a little, we denote

$$\Pi_{\underline{v}} = \Pi^{tp}_{\underline{\underline{X}}_{\underline{v}}}, \quad \Pi^{\pm}_{\underline{v}} = \Pi^{tp}_{\underline{X}_{\underline{v}}}, \quad \Pi^{cor}_{\underline{v}} = \Pi^{tp}_{C_{\underline{v}}}$$

as well as $G_{\underline{v}} = G_K$. Thus we have inclusions

$$\Pi_{\underline{v}} \subset \Pi^{\pm}_{\underline{v}} \subset \Pi^{cor}_{\underline{v}}.$$

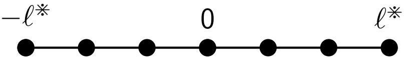
We make similar changes for geometric fundamental groups.

Evaluation - motivation

Our next goal will be to evaluate $\underline{\underline{\theta}}$ at various translations of the evaluation point $\mu_-(\underline{\underline{\ddot{Y}}})$.
It requires choosing a decomposition group for every evaluation point.
This would induce (independent) conjugacy indeterminacies at each point.
However, to achieve the compatibility with Kummer isomorphisms we need to reduce it to a one "diagonal" indeterminacy.
This is obtained by choosing decomposition groups more carefully and analysing how the construction behaves under various conjugation actions.

Subgraphs ([IUT2], §2, 2.1.1)

Write $\Gamma_{\underline{X}}$ for the dual graph of the special fibre of $\underline{X}$. Let $\Gamma^{\blacktriangleright}_{\underline{X}} \subset \Gamma_{\underline{X}}$ be the following subgraph:



Also, let $\Gamma^{\bullet t}_{\underline{X}}$ be the subgraph consisting of only one vertex with label t (thus $t \in \{-\ell^{*}, \ldots, 0, \ldots \ell^{*}\}$, where $\ell^{*} = (\ell - 1)/2$). We lift these subgraphs to appropriate "ι-invariant" subgraphs

$$\Gamma^{\bullet t}_{\underline{\underline{X}}} \subset \Gamma^{\blacktriangleright}_{\underline{\underline{X}}} \subset \Gamma_{\underline{\underline{X}}}, \quad \Gamma^{\bullet t}_{\underline{\underline{\ddot{Y}}}} \subset \Gamma^{\blacktriangleright}_{\underline{\underline{\ddot{Y}}}} \subset \Gamma_{\underline{\underline{\ddot{Y}}}}.$$

We have decomposition groups associated to this subgraphs

$$\Pi_{\underline{v}^{\bullet t}} \subset \Pi_{\underline{v}^{\blacktriangleright}} \subset \Pi_{\underline{v}}.$$

We also define

$$\Pi_{\underline{v}^{\ddot{\blacktriangleright}}} = \Pi_{\underline{v}^{\blacktriangleright}} \cap \Pi^{tp}_{\underline{\underline{\ddot{Y}}}}.$$

Subgraphs (2)

Let $I_t \subset \Pi_{\underline{v}}$ be an inertia group corresponding to the label t.
This determines (uniquely) a decomposition group $D_t \subset \Pi_{\underline{v}\ddot{\blacktriangleright}}$. It also determines a decomposition group

$$D_{t,\mu_-} \subset \Pi_{\underline{v}\ddot{\blacktriangleright}}$$

corresponding to μ_- translation of the cusp with label t (up to $\Delta^{\pm}_{\underline{v}}$-conjugation).
Moreover, this construction is compatible with $\widehat{\Delta}^{\pm}_{\underline{v}}$-conjugation.
Furthermore, if $I_t \subset \Pi_{\underline{v}\bullet t}$, then the construction of D_{t,μ_-} is compatible with respect to conjugation by $\widehat{\Delta}^{cor}_{\underline{v}}$.
(here we are simplifying a little, more precisely one needs to start with $I^{\delta}_t \subset \Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}$, where $\delta, \gamma \in \widehat{\Delta}^{\pm}_{\underline{v}}$).

Evaluation ([IUT2], §2, 2.5)

By restricting $\underline{\underline{\theta}}^{\iota}_{env}$ to the decomposition group $D_{t,\mu_-} \subset \Pi_{\underline{v}}^{\blacktriangleright}$ and applying induced isomorphism $D_{t,\mu_-} \cong G_{\underline{v}}(\mathbb{M}_*)$ we obtain classes

$$\underline{\underline{\theta}}^{t}_{env}(\mathbb{M}_*) \subset H^1(G_{\underline{v}}(\mathbb{M}_*), \Pi_{\mu}(\mathbb{M}_*))$$

which correspond to the values $\mu_{2\ell}\underline{\underline{q}}^{t^2}$ (similarly for the ∞-version). Then, the sets $\underline{\underline{\theta}}^{t}_{env}(\mathbb{M}_*)$ depend only on the value $|t|$, thus we will write

$$\underline{\underline{\theta}}^{|t|}_{env}(\mathbb{M}_*) \subset {}_{\infty}\underline{\underline{\theta}}^{|t|}_{env}(\mathbb{M}_*),$$

here $|t| \in \{0, 1, \ldots \ell^*\}$.

Evaluation (2)

Therefore, we obtain a collection of values

$$\{\underline{\underline{\theta}}^{|t|}_{env}(\mathbb{M}_*)\}_{|t|}$$

for all $|t| \in \{0, 1, \ldots \ell^*\}$ corresponding to $\zeta_{2\ell}\underline{\underline{q}}^{j^2}$ for $0 \leq j \leq \ell^*$, here $\zeta_{2\ell}$ are 2ℓ-roots of unity (potentially different $\zeta_{2\ell}$ for different j).
Next, we want to construct a "container" for this values which will be of the form "$\mathcal{O}^\triangleright$", thus equipped with various valuation maps $\mathcal{O}^\triangleright \to \mathbb{Q}_{\geq 0}$.

Constant monoids

Recall the following construction: from the topological group $G_{\underline{v}}$ we may construct a cyclotome $\mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}})$ and a surjection

$$H^1(G_{\underline{v}}, \mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}})) \twoheadrightarrow \widehat{\mathbb{Z}}.$$

Thus, by taking the preimage of $\mathbb{Z} \subset \widehat{\mathbb{Z}}$ and a colimit over all open subgroups $H \subset G_{\underline{v}}$ we obtain a topological monoid with an action of $G_{\underline{v}}$:

$$G_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}(G_{\underline{v}}),$$

isomorphic to $G_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}_{K^{\mathrm{alg}}}$.

Constant monoids (2)

Moreover, if $G_{\underline{v}} \curvearrowright M$ is an isomorph of $G_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}_{K^{\mathrm{alg}}}$, then writing

$$\mu_{\widehat{\mathbb{Z}}}(M) = lim_n M[n] \quad \text{("Tate module" of } M)$$

we may construct (from the pair $G_{\underline{v}} \curvearrowright M$) an isomorphism $\mu_{\widehat{\mathbb{Z}}}(M) \cong \mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}})$ which induces an isomorphism

$$G_{\underline{v}} \curvearrowright M \cong G_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}(G_{\underline{v}})$$

after passing to cohomology via the Kummer map. The isomorphism

$$\mu_{\widehat{\mathbb{Z}}}(M) \cong \mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}})$$

is also called a cyclotomic rigidity isomorphism.

Constant monoids (3)

Using anabelian geometry, from a topological group $\Pi_{\underline{v}}$ one can construct a natural isomorphism

$$\ell\Delta_{\Theta}(\Pi_{\underline{v}}) \cong \mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}}).$$

Therefore, pulling back by this isomorphism we obtain a monoid

$$\Psi_{cns}(\mathbb{M}_*) \subset {}_{\infty}H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}_*), \Pi_{\mu}(\mathbb{M}_*)).$$

(called a mono-theta constant monoid) corresponding to "$\mathcal{O}^{\triangleright}_{K^{\mathrm{alg}}}$". However, note that the construction of the constant monoid does not have the multiradiality property. Roughly speaking, that is because nontrivial automorphisms of $G_{\underline{v}} \curvearrowright \mathcal{O}^{\times}$ do not lift to automorphisms of $G_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}$. Making this part of construction "multiradial" requires considering "log-links" and it is done in IUT3.

Gaussian monoids ([IUT2], §3, 3.5)

Recall that $\Delta_C(\mathbb{M}_*)/\Delta_{\underline{X}}(\mathbb{M}_*) = \mathbb{F}_\ell^{\rtimes\pm}$ acts on the set of cusps of $\underline{X}$. Thus, denoting by t, t' labels of cusps of $\underline{X}$, we have an isomorphism

$$(G_{\underline{v}}(\mathbb{M}_*) \curvearrowright \Psi_{cns}(\mathbb{M}_*))_t \cong (G_{\underline{v}}(\mathbb{M}_*) \curvearrowright \Psi_{cns}(\mathbb{M}_*))_{t'}$$

between labeled copies of constant monoids. We identify labels t and $-t$ by means of a label $|t|$. (thus $|t| \in \{0, \dots, \ell^*\} = |\mathbb{F}_\ell|$). Also we write

$$\Psi_{cns}(\mathbb{M}_*))_{\langle |F_\ell| \rangle} \subset \prod_{|t|} \Psi_{cns}(\mathbb{M}_*)_t$$

for the "diagonal" of this product, similarly for $\langle \mathbb{F}_\ell^* \rangle = \{1, \dots, \ell^*\}$.

Gaussian monoids (2)

Define a value profile (i.e., collection $(\zeta_{2\ell}\underline{\underline{q}}^{1^2}, \ldots, \zeta_{2\ell}\underline{\underline{q}}^{(\ell^*)^2})$ of values) as

$$\underline{\underline{\theta}}_{env}^{\mathbb{F}_\ell^{\circledast}}(\mathbb{M}_*) = \prod_{|t| \in \mathbb{F}_\ell^{\circledast}} \underline{\underline{\theta}}_{env}^{|t|}(\mathbb{M}_*) \subset \prod_{|t| \in \mathbb{F}_\ell^{\circledast}} \Psi_{cns}(\mathbb{M}_*)_{|t|}.$$

Then, we define the Gaussian monoid (i.e., $\mathcal{O}^\times \cdot (\zeta_{2\ell}\underline{\underline{q}}^{1^2}, \ldots, \zeta_{2\ell}\underline{\underline{q}}^{(\ell^*)^2})$) as

$$\Psi_{gau}(\mathbb{M}_*) = \{\Psi_{cns}^{\times}(\mathbb{M}_*)_{\langle \mathbb{F}_\ell^{\circledast} \rangle} \cdot \xi^{\mathbb{N}}\}_\xi$$

where ξ runs through all value profiles. Note that we treat the Gaussian monoid as a subset of (a product of) constant monoids. Thus we do not have multiradiality property for the Gaussian monoid.

Globalization ([IUT2], §4)

We will very briefly describe how Gaussian monoid fits into a global object, namely a Gaussian (realified) Frobenioid. In fact, this construction is mostly formal, the nontrival part is played by bad local places.
First, recall that realified Frobenioid is essentially a collection of monoids $\mathbb{R}_{\geq 0}$ together with the "product formula". Moreover, to obtain additionally an $\mathcal{F}^{\Vdash}$-prime strip we need to specify group of units "$\mathcal{O}^{\times}$". Thus we need to define these monoids at good nonarchimedean(as well as at archimedean places, which we completely omit in this discussion).
At good places, we have a notion of semisimplicifaction

$$\Psi^{ss}_{cns}(G_{\underline{v}}) = \Psi^{\times}_{cns}(G_{\underline{v}}) \times \mathbb{R}_{\geq 0}(G_{\underline{v}})$$

where

$$\mathbb{R}_{\geq 0}(G_{\underline{v}}) = (\Psi_{cns}(G_{\underline{v}})/\Psi^{\times}_{cns}(G_{\underline{v}}))^{rlf}$$

Globalization (2)

Then, roughly speaking, we use $\mathbb{R}_{\geq 0}(G_{\underline{v}})$ to extend the Gaussian monoids to various realified Frobenioids:

- group theoretic $\mathcal{D}^{\Vdash}_{gau}(\mathfrak{D}^{\vdash}_{\succ})$ ([IUT2], §4, 4.5),
- Frobenioid theoretic $\mathcal{C}^{\Vdash}_{gau}(\mathcal{HT}^{\Theta})$ ([IUT2], §4, 4.6).

Then, the Frobeniod $\mathcal{C}^{\Vdash}_{gau}(\mathcal{HT}^{\Theta})$, appropriately "enhanced" with the unit part "$\mathcal{O}^{\times}$" forms a $\mathcal{F}^{\Vdash}$-prime strip denoted by $\mathfrak{F}^{\Vdash}_{gau}$ ([IUT2], §4, 4.10).

Summary and what is next?

To summarise, we have constructed values

$$\underline{\underline{q}}^{j^2} \subset \mathcal{O}^{\triangleright}$$

by evaluating theta function at various points. However, this construction is not multiradial (i.e., not compatible with $G \curvearrowright \mathcal{O}^{\times\mu}$) since we used constant monoids as containers of these values (but recall that the first part of construction, namely theta monoids, was multiradial).
To solve this problem, we will consider various diagrams of the form:

$$\underline{\underline{q}}^{j^2} \subset \mathcal{O}^{\triangleright} \subset (\mathcal{O}^{\triangleright})^{gp} \cong \mathcal{O}^{\times\mu}$$

where the last isomorphism is given by the p-adic logarithm ($\rightsquigarrow$ IUT3).

End of the talk

Thank you for your attention!